

# Note on the Painlevé V tau-functions

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## Abstract

We study some properties of tau-functions of an isomonodromic deformation leading to the fifth Painlevé equation. In particular, here is given an elementary proof of Miwa's formula for the logarithmic differential of a tau-function.

## 1 Introduction

This work is an addition to the article [1], where we studied some properties of the Malgrange isomonodromic deformation of a linear differential  $(2 \times 2)$ -system defined on the Riemann sphere  $\overline{\mathbb{C}}$  and having at most two irregular singularities of Poincaré rank one. Here we consider a particular case of such a system:

$$\frac{dy}{dz} = \left( \frac{B_0^0}{z} + \frac{B_1^0}{z - t_0} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) y, \quad (1)$$

where  $y(z) \in \mathbb{C}^2$  and  $B_0^0, B_1^0$  are  $(2 \times 2)$ -matrices. This system has two Fuchsian singular points  $z = 0, z = t_0 \in \mathbb{C} \setminus \{0\}$  and one *non-resonant* irregular singularity  $z = \infty$  of Poincaré rank one.

As known (see [10, §§10,11] or [3, §21]), in a neighbourhood of (non-resonant) irregular singularity  $z = \infty$  the system (1) is formally equivalent to a system

$$\frac{d\tilde{y}}{dz} = \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{A}{z} \right) \tilde{y},$$

where  $A$  is a diagonal  $(2 \times 2)$ -matrix. This means that there is an invertible matrix formal Taylor series  $\hat{F}(z)$  in  $1/z$  beginning with the identity matrix such that these two systems are connected by means of the transformation  $y = \hat{F}(z)\tilde{y}$  (and such a series  $\hat{F}$  is unique). Thus, the system (1) possesses a uniquely determined formal fundamental matrix  $\hat{Y}(z)$  of the form

$$\hat{Y}(z) = \hat{F}(z) z^A e^{\text{diag}(z,0)}.$$

According to Sibuya's sectorial normalization theorem (see [3, Th. 21.13, Prop. 21.17]), a punctured neighbourhood of the point  $z = \infty$  is covered by two sectors

$$S_1 = \left\{ \frac{\pi}{2} - \varepsilon < \arg z < \frac{3\pi}{2} + \varepsilon, \quad |z| > R \right\}, \quad S_2 = \left\{ -\frac{\pi}{2} - \varepsilon < \arg z < \frac{\pi}{2} + \varepsilon, \quad |z| > R \right\},$$

with a sufficiently small  $\varepsilon > 0$  and sufficiently large  $R > 0$ , such that in each  $S_i$  there exists a *unique* actual fundamental matrix

$$Y_i(z) = F_i(z) z^A e^{\text{diag}(z,0)}$$

of the system (1) whose factor  $F_i(z)$  has the asymptotic expansion  $\hat{F}(z)$  in  $S_i$ .

The intersection  $S_1 \cap S_2$  is a union of two sectors  $\Sigma_1, \Sigma_2$ ,

$$\Sigma_1 = \left\{ \frac{\pi}{2} - \varepsilon < \arg z < \frac{\pi}{2} + \varepsilon, \quad |z| > R \right\}, \quad \Sigma_2 = \left\{ \frac{3\pi}{2} - \varepsilon < \arg z < \frac{3\pi}{2} + \varepsilon, \quad |z| > R \right\}.$$

In the sector  $\Sigma_1$  the fundamental matrices  $Y_1$  and  $Y_2$  necessarily differ by a constant invertible matrix:

$$Y_2(z) = Y_1(z)C_1, \quad C_1 \in \text{GL}(2, \mathbb{C}), \quad z \in \Sigma_1.$$

In the sector  $\Sigma_2$ , one similarly has

$$Y_1(z) = Y_2(z)C_2, \quad C_2 \in \text{GL}(2, \mathbb{C}), \quad z \in \Sigma_2.$$

The matrices  $C_1, C_2$  are called (Sibuya's) Stokes matrices of the system (1) at the non-resonant irregular singular point  $z = \infty$  (more precisely, the second Stokes matrix is  $C_2 e^{-2\pi i A}$ ).

Further we will focus on isomonodromic deformations of the system (1) which are closely related to the fifth Painlevé equation. According to M. Jimbo [4] such an isomonodromic deformation is a family

$$\frac{dy}{dz} = \left( \frac{B_0(t)}{z} + \frac{B_1(t)}{z-t} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) y, \quad B_{0,1}(t_0) = B_{0,1}^0, \quad (2)$$

of differential systems holomorphically depending on the parameter  $t \in D(t_0)$ , where  $D(t_0) \subset \mathbb{C}$  is a neighbourhood of the point  $t_0$  and the matrix functions  $B_0(t), B_1(t)$  are determined by the integrability condition  $d\Omega = \Omega \wedge \Omega$  for the matrix meromorphic differential 1-form

$$\Omega = \left( \frac{B_0(t)}{z} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) dz + \frac{B_1(t)}{z-t} d(z-t)$$

on the space  $\overline{\mathbb{C}} \times D(t_0)$ . The isomonodromy of the family (2) means that a part of its *monodromy data* is independent of  $t$ . This part consists of the monodromy matrices corresponding to some fundamental matrix  $Y(z, t)$  of (2), Stokes matrices at the infinity and the connection matrix between  $Y$  and  $Y_1$ .

Assuming the eigenvalues of  $B_0(t) = (b_0^{ij}(t))$ ,  $B_1(t) = (b_1^{ij}(t))$  to be  $\pm \frac{1}{2} \theta_0$ ,  $\pm \frac{1}{2} \theta_1 \notin \frac{1}{2} \mathbb{Z}$  (they do not depend on  $t$ ) and  $B_0(t) + B_1(t)$  equivalent to  $\text{diag}(-\frac{1}{2} \theta_\infty, \frac{1}{2} \theta_\infty)$ , one has the function

$$u(t) = \frac{b_1^{12}(t) \left( b_0^{11}(t) + \frac{1}{2} \theta_0 \right)}{b_0^{12}(t) \left( b_1^{11}(t) + \frac{1}{2} \theta_1 \right)}$$

to satisfy the fifth Painlevé equation

$$\frac{d^2 u}{dt^2} = \left( \frac{1}{2u} + \frac{1}{u-1} \right) \left( \frac{du}{dt} \right)^2 - \frac{1}{t} \frac{du}{dt} + \frac{(u-1)^2}{t^2} \left( \alpha u + \frac{\beta}{u} \right) + \frac{\gamma u}{t} + \frac{\delta u(u+1)}{u-1},$$

where

$$\alpha = \frac{1}{8}(\theta_0 - \theta_1 + \theta_\infty)^2, \quad \beta = -\frac{1}{8}(\theta_0 - \theta_1 - \theta_\infty)^2, \quad \gamma = 1 - \theta_0 - \theta_1, \quad \delta = -\frac{1}{2}.$$

There is also a geometric approach of B. Malgrange [6] to the isomonodromic deformation of the system (1) we referred to in [1], but here we do not immerse in details of that approach in view of the sufficiency of an analytic language for the present note.

According to the Miwa–Malgrange–Helminck–Palmer theorem (see [7], [6, §3], [2] or [9, §3]) the matrix functions  $B_0(t)$ ,  $B_1(t)$  holomorphic in  $D(t_0)$  can be extended meromorphically to the universal cover  $\mathcal{D} \cong \mathbb{C}$  of the set  $\mathbb{C} \setminus \{0\}$  of locations of the pole  $t_0$ . The set  $\Theta \subset \mathcal{D}$  of the poles of the extended matrix functions  $B_0$ ,  $B_1$  (which may be empty) is usually called the *Malgrange  $\Theta$ -divisor* of the family (2). For  $t^* \in \Theta$ , a *local  $\tau$ -function* of (2) is any holomorphic in  $D(t^*)$  function  $\tau^*$  such that  $\tau^*(t^*) = 0$ . There exists a function  $\tau$  (called a *global  $\tau$ -function* of the isomonodromic deformation (2) or of the fifth Painlevé equation) holomorphic on the whole space  $\mathcal{D}$  whose zero set coincides with  $\Theta$ . Thus, the set  $\Theta$  has no limit points in  $\mathcal{D}$ . As follows from the results of [1], all the zeros of this  $\tau$ -function are simple (at least in the case of the irreducible monodromy of (2)). Here we give an elementary proof of Palmer’s theorem [9] concerning a  $\tau$ -function of the isomonodromic deformation (2).

**Theorem 1.** *A  $\tau$ -function of the fifth Painlevé equation satisfies the equality*

$$d \ln \tau(t) = \frac{1}{2} \operatorname{res}_{z=t} \operatorname{tr} \left( B(z, t) \right)^2 dt,$$

where  $B(z, t)$  denotes the coefficient matrix of the family (2).

The above formula is also referred to as a definition of a  $\tau$ -function (see [5], [4]). Having this definition T. Miwa [7] has proved the analyticity of such a  $\tau$ -function on  $\mathcal{D}$  (then its zeros are *a priori* included in  $\Theta$ ).

Another definition of a  $\tau$ -function of the Painlevé V (and of the other Painlevé equations) comes from a Hamiltonian form of this equation. As K. Okamoto has shown [8], there is a function  $\tau$  holomorphic on  $\mathcal{D}$  whose logarithmic derivative is equal to a Hamiltonian along a solution (this  $\tau$ -function depends on a solution, its zeros are included in the set of poles of a solution).

## 2 Proof of Theorem 1

Consider a point  $t^* \in \Theta$  of the Malgrange  $\Theta$ -divisor of (2). Assuming that  $t^* \neq -1$  we make the transformation  $\xi = 1/(z+1)$  of the independent variable<sup>1</sup> and come from (2) to the isomonodromic family

$$\frac{d\tilde{y}}{d\xi} = \left( \frac{B_0(t(s))}{\xi-1} + \frac{B_1(t(s))}{\xi-s} - \frac{1}{\xi^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{B_0(t(s)) + B_1(t(s))}{\xi} \right) \tilde{y}, \quad \tilde{y}(\xi) = y(z(\xi)), \quad (3)$$

depending on the parameter

$$s = \frac{1}{t+1} \quad \left( \Rightarrow t = \frac{1-s}{s} \right),$$

with the Fuchsian singular points  $\xi = 1$ ,  $\xi = s$  and non-resonant irregular singularity  $\xi = 0$  of Poincaré rank 1. The infinity is a non-singular point of (3), therefore this family is a particular case of those considered in the paper [1]. Using some properties of such families obtained in that paper we will prove Miwa’s formula for a local  $\tau$ -function of (3) in a neighbourhood of the point  $s^* = 1/(t^* + 1) \in s(\Theta)$ . Namely, the following statement holds whose proof is presented in the next section.

**Lemma 1.** *There is a local  $\tau$ -function  $\tilde{\tau}(s)$  of the family (3) near  $s^*$  such that*

$$d \ln \tilde{\tau}(s) = \frac{1}{2} \operatorname{res}_{\xi=s} \operatorname{tr} \left( \tilde{B}(\xi, s) \right)^2 ds,$$

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<sup>1</sup>If  $t^* = -1$  one makes a transformation  $\xi = 1/(z-c)$ ,  $c \neq -1$ .

where  $\tilde{B}(\xi, s)$  is the coefficient matrix of this family.

Having Lemma 1 we conclude for the local  $\tau$ -function  $\tilde{\tau}(s(t))$  of (2) near  $t^*$  that

$$d \ln \tilde{\tau}(s(t)) = \frac{1}{2} \text{res}_{\xi=s} \text{tr} \left( \tilde{B}(\xi, s) \right)^2 (-s^2) dt.$$

To connect  $\text{res}_{\xi=s} \text{tr} \left( \tilde{B}(\xi, s) \right)^2 (-s^2)$  and  $\text{res}_{z=t} \text{tr} \left( B(z, t) \right)^2$ , let us compute the both. Since

$$\text{res}_{z=t} \left( B(z, t) \right)^2 = \frac{B_0 B_1 + B_1 B_0}{t} + B_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B_1,$$

one has

$$\frac{1}{2} \text{res}_{z=t} \text{tr} \left( B(z, t) \right)^2 = \text{tr} \frac{B_0 B_1}{t} + \text{tr} \left( B_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right).$$

In a similar way,

$$\text{res}_{\xi=s} \left( \tilde{B}(\xi, s) \right)^2 = \frac{B_0 B_1 + B_1 B_0}{s-1} - \frac{B_0 B_1 + B_1 B_0 + 2B_1^2}{s} - \frac{1}{s^2} \left( B_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B_1 \right),$$

and

$$\begin{aligned} \frac{1}{2} \text{res}_{\xi=s} \text{tr} \left( \tilde{B}(\xi, s) \right)^2 (-s^2) &= \frac{s^2}{1-s} \text{tr}(B_0 B_1) + s \text{tr}(B_0 B_1) + s \text{tr} B_1^2 + \text{tr} \left( B_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) = \\ &= \frac{s}{1-s} \text{tr}(B_0 B_1) + s \text{tr} B_1^2 + \text{tr} \left( B_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) = \\ &= \text{tr} \frac{B_0 B_1}{t} + \text{tr} \left( B_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) + \text{tr} \frac{B_1^2}{t+1}. \end{aligned}$$

Therefore,

$$d \ln \tilde{\tau}(s(t)) = \frac{1}{2} \text{res}_{z=t} \text{tr} \left( B(z, t) \right)^2 dt + \text{tr} \frac{B_1^2}{t+1} dt.$$

Denoting now  $B_1(t) = (b_{ij}(t))$  we have

$$\begin{aligned} \text{tr} B_1^2 &= (b_{11})^2 + 2b_{12}b_{21} + (b_{22})^2 = (b_{11} + b_{22})^2 - 2(b_{11}b_{22} - b_{12}b_{21}) = \\ &= (\text{tr} B_1)^2 - 2 \det B_1 = \theta_1^2/2 = \text{const} \end{aligned}$$

(recall that the eigenvalues of the matrix  $B_1(t)$  are  $\pm \theta_1/2$  not depending on  $t$ ). Hence,

$$d \ln \frac{\tilde{\tau}(s(t))}{(t+1)^{\theta_1^2/2}} = \frac{1}{2} \text{res}_{z=t} \text{tr} \left( B(z, t) \right)^2 dt.$$

Thus, near a point  $t^* \in \Theta$  we have the formula

$$d \ln \tau^*(t) = \frac{1}{2} \text{res}_{z=t} \text{tr} \left( B(z, t) \right)^2 dt \quad (4)$$

for the local  $\tau$ -function  $\tau^*(t) = \tilde{\tau}(s(t))/(t+1)^{\theta_1^2/2}$  of (2).

Further we use standard reasonings to come from the local  $\tau$ -functions to a global one and finish the proof of the theorem. Consider a covering  $\{U_\alpha\}$  of the deformation space  $\mathcal{D}$  such that

for every  $U_\alpha$  there is a holomorphic function  $\tau_\alpha(t)$  satisfying the equality (4) and non-vanishing in  $U_\alpha$  if  $U_\alpha \cap \Theta = \emptyset$ . In every non-empty intersection  $U_\alpha \cap U_\beta$  one has

$$\tau_\alpha(t) = c_{\alpha\beta} \tau_\beta(t), \quad (5)$$

where  $c_{\alpha\beta} = \text{const} \neq 0$ , since  $d \ln c_{\alpha\beta} = d \ln \tau_\alpha - d \ln \tau_\beta = 0$ . The equalities (5) imply

$$c_{\alpha\beta} c_{\beta\gamma} = c_{\alpha\gamma} \quad (6)$$

for non-empty intersections  $U_\alpha \cap U_\beta \cap U_\gamma$ .

Let us fix logarithms  $l_{\alpha\beta} = \ln c_{\alpha\beta}$  in such a way that  $l_{\alpha\beta} = -l_{\beta\alpha}$ . Then, as follows from (6),

$$l_{\alpha\beta} - l_{\alpha\gamma} + l_{\beta\gamma} = 2\pi i l_{\alpha\beta\gamma},$$

where the set of numbers  $l_{\alpha\beta\gamma} \in \mathbb{Z}$  defines an element of the Čech cohomology group  $H^2(\mathcal{D}, \mathbb{Z})$ . Since  $\mathcal{D} \cong \mathbb{C}$ , one has  $H^2(\mathcal{D}, \mathbb{Z}) = 0$ , hence there is a set  $\{l'_{\alpha\beta}\} \subset \mathbb{Z}$  such that

$$l_{\alpha\beta\gamma} = l'_{\alpha\beta} - l'_{\alpha\gamma} + l'_{\beta\gamma} \quad \text{for } U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset.$$

Therefore

$$(l_{\alpha\beta} - 2\pi i l'_{\alpha\beta}) - (l_{\alpha\gamma} - 2\pi i l'_{\alpha\gamma}) + (l_{\beta\gamma} - 2\pi i l'_{\beta\gamma}) = 0,$$

and the set of numbers  $\lambda_{\alpha\beta} = l_{\alpha\beta} - 2\pi i l'_{\alpha\beta} \in \mathbb{C}$  defines an element of the Čech cohomology group  $H^1(\mathcal{D}, \mathbb{C})$ . Since the latter is trivial, there is a set  $\{\lambda_\alpha\} \subset \mathbb{C}$  such that

$$\lambda_{\alpha\beta} = \lambda_\alpha - \lambda_\beta \quad \text{for } U_\alpha \cap U_\beta \neq \emptyset.$$

Thus, a set  $\{c_\alpha = e^{\lambda_\alpha}\}$  of non-zero constants is such that

$$c_{\alpha\beta} = e^{l_{\alpha\beta}} = e^{\lambda_{\alpha\beta}} = c_\alpha c_\beta^{-1}$$

in every non-empty intersection  $U_\alpha \cap U_\beta$ . Therefore we have a global function  $\tau(t)$  holomorphic on  $\mathcal{D}$  and equal to  $c_\alpha^{-1} \tau_\alpha(t)$  in every  $U_\alpha$ , hence satisfying (4).

### 3 Proof of Lemma 1

Consider a point  $s^* \in s(\Theta)$ . Though the family (3) is not defined for this value of the parameter, one can construct an auxiliary linear meromorphic  $(2 \times 2)$ -system

$$\frac{dw}{d\xi} = A^*(\xi) w, \quad A^*(\xi) = \frac{A_{01}^*}{\xi} + \frac{A_{02}^*}{\xi^2} + \frac{A_1^*}{\xi - 1} + \frac{A_2^*}{\xi - s^*}, \quad (7)$$

with irregular non-resonant singular point  $\xi = 0$  of Poincaré rank 1 and Fuchsian singular points  $1, s^*$ . In a neighbourhood of zero this system is holomorphically equivalent to the image of the initial system (1) under the change of variable  $\xi = 1/(z + 1)$ , has the same monodromy matrices as the latter, but it has an *apparent* Fuchsian singularity at the infinity (*i. e.*, the monodromy at this point is trivial).

We will use the following facts explained in [1]:

- the auxiliary system (7) is included into a (Malgrange) isomonodromic family

$$\frac{dw}{d\xi} = \left( \frac{A_{01}(s)}{\xi} + \frac{A_{02}(s)}{\xi^2} + \frac{A_1(s)}{\xi-1} + \frac{A_2(s)}{\xi-s} \right) w \quad (8)$$

whose isomonodromic fundamental matrix  $W(\xi, s)$  near the point  $\xi = \infty$  has the form

$$W(\xi, s) = U(\xi, s)\xi^K, \quad U(\xi, s) = I + U_1(s)\frac{1}{\xi} + U_2(s)\frac{1}{\xi^2} + \dots, \quad (9)$$

where  $K = \text{diag}(-1, 1)$  and  $\frac{dU_1(s)}{ds} = -A_2(s)$ ;

- the upper right element  $u_1(s)$  of the matrix  $U_1(s)$  vanishes at the point  $s = s^*$  and is not equal to zero identically, hence it is a local  $\tau$ -function of the family (3) (and  $\frac{du_1}{ds}(s^*) \neq 0$  if the monodromy of (3) is irreducible);
- for  $s \neq s^*$  the family (3) is meromorphically equivalent to (8) *via* a gauge transformation  $\tilde{y} = \Gamma_1(\xi, s)w$ , where

$$\Gamma_1(\xi, s) = \begin{pmatrix} \frac{f(s)}{u_1(s)} + \xi & -u_1(s) \\ \frac{1}{u_1(s)} & 0 \end{pmatrix},$$

for some function  $f$  holomorphic at the point  $s = s^*$ .

The coefficient matrices  $\tilde{B}(\xi, s)$  and  $A(\xi, s)$  of the families (3) and (8) respectively are connected by the equality

$$\tilde{B}(\xi, s) = \frac{\partial \Gamma_1}{\partial \xi} \Gamma_1^{-1} + \Gamma_1 A(\xi, s) \Gamma_1^{-1},$$

therefore

$$\begin{aligned} \left( \tilde{B}(\xi, s) \right)^2 &= \left( \frac{\partial \Gamma_1}{\partial \xi} \Gamma_1^{-1} \right)^2 + \frac{\partial \Gamma_1}{\partial \xi} A(\xi, s) \Gamma_1^{-1} + \\ &+ \Gamma_1 A(\xi, s) \Gamma_1^{-1} \frac{\partial \Gamma_1}{\partial \xi} \Gamma_1^{-1} + \Gamma_1 \left( A(\xi, s) \right)^2 \Gamma_1^{-1}. \end{aligned}$$

As follows from the form of the matrix  $\Gamma_1(\xi, s)$ , the product  $(\frac{\partial}{\partial \xi} \Gamma_1) \Gamma_1^{-1}$  is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & u_1 \\ -\frac{1}{u_1} & * \end{pmatrix} = \begin{pmatrix} 0 & u_1 \\ 0 & 0 \end{pmatrix},$$

hence its square is the zero matrix and

$$\text{tr} \left( \tilde{B}(\xi, s) \right)^2 = 2 \text{tr} \left( \Gamma_1^{-1} \frac{\partial \Gamma_1}{\partial \xi} A(\xi, s) \right) + \text{tr} \left( A(\xi, s) \right)^2.$$

Since

$$\Gamma_1^{-1} \frac{\partial \Gamma_1}{\partial \xi} = \begin{pmatrix} 0 & 0 \\ -\frac{1}{u_1} & 0 \end{pmatrix},$$

one has

$$\text{tr} \left( \tilde{B}(\xi, s) \right)^2 = -2 \frac{a(\xi, s)}{u_1(s)} + \text{tr} \left( A(\xi, s) \right)^2,$$

where  $a(\xi, s)$  is the upper right element of the matrix  $A(\xi, s)$ . Thus,

$$\text{res}_{\xi=s} \text{tr} \left( \tilde{B}(\xi, s) \right)^2 ds = -2 \frac{\text{res}_{\xi=s} a(\xi, s)}{u_1(s)} ds + \text{res}_{\xi=s} \text{tr} \left( A(\xi, s) \right)^2 ds.$$

As  $\text{res}_{\xi=s}a(\xi, s)$  is equal to the upper right element of  $A_2(s)$ , which is  $-\frac{du_1}{ds}$ , one has

$$\text{res}_{\xi=s}\text{tr}\left(\tilde{B}(\xi, s)\right)^2 ds = 2 d \ln u_1(s) + \text{res}_{\xi=s}\text{tr}\left(A(\xi, s)\right)^2 ds.$$

The differential 1-form  $\text{res}_{\xi=s}\text{tr}\left(A(\xi, s)\right)^2 ds$  is closed and holomorphic in a neighbourhood  $D(s^*)$  of the point  $s = s^*$ , hence there is a function  $f^*$  holomorphic and non-vanishing in  $D(s^*)$  such that  $d \ln f^*(s) = \frac{1}{2}\text{res}_{\xi=s}\text{tr}\left(A(\xi, s)\right)^2 ds$ . Therefore,

$$\text{res}_{\xi=s}\text{tr}\left(\tilde{B}(\xi, s)\right)^2 ds = 2 d \ln(f^*(s)u_1(s)),$$

which finishes the proof of the lemma, with  $\tilde{\tau}(s) = f^*(s)u_1(s)$ .

**Acknowledgements.** This work is supported by the Russian Foundation for Basic Research (grant no. RFBR-14-01-31145 mol.a) and RF President program for young scientists (grant no. MK-4594.2013.1).

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